THE RIESZ-WIENER AND THE
STEIN-HERZ ESTIMATES OF THE
MAXIMAL FUNCTION

The Hardy-Littlewood maximal function

\[ Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy \]

plays an important role in the study of differentiation, singular integrals, and almost everywhere convergence. A great interest in estimates of rearrangement and distribution of the maximal function started after the classical paper of Hardy and Littlewood (1930). They defined the maximal operator (in the one-dimensional case) and proved boundedness of this operator in \( L_p(\mathbb{R}^1) \) for \( p > 1 \). F. Riesz (1932) proved the rearrangement inequality

\[ (Mf)^*(t) \leq A F^{**}(t) \quad \forall t > 0, f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds. \]  \( \tag{1} \)

N. Wiener (1939) defined the maximal function \( Mf \) in the \( n \)-dimensional case and proved the weak type \( (1,1) \) of maximal operator and also the stronger inequality for the distribution

\[ \lambda |\{ x \in \mathbb{R}^n : Mf(x) > \lambda \}| \leq B \int_{\{ x \in \mathbb{R}^n : |f(x)| > \lambda /2 \}} |f(x)| \, dx \quad \forall \lambda > 0. \]  \( \tag{2} \)

The inverse inequalities to those of Riesz and Wiener

\[ \frac{1}{\lambda} \int_{\{ x \in \mathbb{R}^n : |f(x)| > \lambda /2 \}} |f(x)| \, dx \leq C |\{ x \in \mathbb{R}^n : |f(x)| > \lambda /2 \}| \quad \forall \lambda > 0, \]  \( \tag{3} \)

\[ f^{**}(t) \leq D (Mf)^*(t) \quad \forall t > 0, \]  \( \tag{4} \)

were found by Stein (1968) and Herz (1968).
We discuss the conditions on the measure $d\mu(x) = w(x)dx$ under which similar estimates for the distribution and rearrangement of the weighted maximal function

$$M_w f(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f(y)|w(y)dy$$

are valid. It is proved that the Riesz inequality $(1_w)$ is equivalent to the Wiener inequality $(2_w)$ and that they are true if and only if the maximal operator $M_w$ is of the weak type $(1,1)$. The Stein inequality $(3_w)$ and the Herz inequality $(4_w)$ are valid without any restriction on the measure $w$. In particular,

$$(M_w f)_w^*(t) \approx \frac{1}{t} \int_0^t f_w^*(s) ds \text{ iff the operator } M_w \text{ is of weak type } (1,1).$$

These are my joint results with I. U. Asekritova, N. Ya. Krugljak, and L. E. Persson.

References

