Accuracy of Material Parameter Estimation

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by

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Preface

This work has mainly been carried out at Volvo Aero Corporation in Trollhättan under the supervision of Professor Lars-Erik Lindgren at Luleå University of Technology and Dr. Niklas Järvstråt at Volvo Aero Corporation. It is a result of a joint effort between Volvo Aero Corporation and the Division of Computer Aided Design at Luleå University of Technology. Financially it has been supported by the Swedish National Board for Industrial and Technical Development (NUTEK) within the NFFP programme (Nationella Flygtekniska Forsknings Programmet). I would specially like to thank Lars-Erik as well as my co-supervisor Niklas for their support and interesting discussions. I would also like to thank my colleagues at Volvo Aero Corporation and the division of Computer Aided Design for creating an inspiring work environment.

Trollhättan, December 1999

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Abstract
Trends in military and commercial aircraft usage together with market development has lead to an ever-increasing demand for additional thrust, extended component service life, improved fuel efficiency and reduced weight. This has resulted in engine configurations with high temperatures, high pressures and increased rotor speed, where the engine components have to sustain a more and more severe environment. The components are also subjected to complex load histories during the manufacturing processes, such as welding, forging and heat treatment.

Modern analysis tools, using the finite element method with non-linear material models, provide the means to evaluate the complex thermal and mechanical response of the components. This thesis considers material parameter estimation for non-linear material models, which is an extremely important part for the reliability of finite element analyses.

The work deals with estimating material parameters and quantifies the accuracy in the parameters. The latter enable a total assessment of how the accuracy at different stages in i.e. the life prediction process affects the final estimated life. Thereby it is also possible to use this to estimate the cost and value of improvements in material data in the different stages of the calculation.

The thesis comprises an introductory part and two appended papers. The first paper discusses the influence on the choice of cost-function in the optimisation process for parameters and estimated life. An alternative derivation of the effective stress function method for stress integration is discussed in the second paper.

KEYWORDS
Parameter estimation, accuracy, material models, aircraft engines, life prediction
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Thesis

The thesis comprises of the following papers:


B. Salomonsson, A., Järvstråt, N. *An alternative derivation of the Effective Stress Function method with an extension to transverse recovery*, (Submitted for international publication).
1 Introduction
The licentiate thesis presented here deals with estimating material parameters and quantifying the accuracy of the found parameters.

1.1 Background
The development of the transport sector is constantly striving towards faster, lighter, cheaper and more environmentally friendly transportation. Requirements on the products lead to increased requirements on new technology. The limits for the materials are constantly pushed forward, which in turn raise the demands of reliability in simulations in order to avoid unnecessary testing. Because of this, the aerospace industry is intensively using computations in development, perhaps more than any other industry, Isaksson [1]. Volvo Aero Corporation is a good example of a company where a skilful use of computational tools is a competitive advantage. Most examples here are from the aerospace industry with high temperature applications of mechanical components. Thus, loading situation and material behaviour are complex to model and describe. The necessity of using computer based numerical tools and methods in the development process goes without saying. Computers are becoming faster and it is now possible to incorporate rather complex models to describe and simulate processes.

To achieve accuracy more complex models has to be used to predict quantities such as stress distribution, rate of crack propagation, safe operating life and residual stresses. In all simulations material data is required. Thus, reliable material data or material parameters are of utmost importance. The identification of these parameters is normally based on experimental tests. Great sums are yearly spent on conventional testing in order to produce the data. It is important to have systematic, efficient and accurate methods for determining material parameters for complex models, in e.g. visco-plasticity or crack propagation.

The improved methods for parameter identification treated in this thesis lead to:
• A more systematic procedure as the process is computer based. That is, the estimation process becomes less dependent on the evaluator.
• More efficient simulation procedures for identification, i.e. the evaluation is faster.
• A higher reliability as it is possible to quantify the accuracy of the design data.
• More cost-effective testing as it is possible to quantify the effect of reduced uncertainty in test data on the reliability of design data.
• More information can be extracted from the experiments and thereby minimise the number of experiments that have to be performed when inverse modelling is used.

1.2 Aim and scope of thesis
This work is part of a project with the object to develop a methodology for evaluation of materials testing. Specifically, we aim to generate and classify design data for specific materials. The classification involves quantified errors in the parameters, and estimated region of validity (in strains, temperatures etc) where the data has a satisfactory precision. The process of parameter estimation is made efficient and systematic as the methodology is embodied in a computer software. The aim can be reformulated into the research question:
What is the best methodology for determination of material parameters and how can the effect of uncertainties in parameters be evaluated?

A computational tool will be developed and demonstrated on some specific material models and materials. The finite element method in conjunction with models for material behaviour and optimisation methods will be used in the parameter evaluation process (i.e. inverse modelling). The work is covering material testing, mechanics of solid materials and numerical methods.
The scope of the work presented in this thesis has been to implement methods for parameter estimation and study two areas of parameter estimation analysis: the influence of different cost-functions when the model is not sufficient to describe the material behaviour, and the integration techniques for the stress integration.

The summary of the thesis is organised as follow: An introduction to gas turbine technology is given in Chapter 2. A general description of material models for the high temperature regime is presented in Chapter 3. Here, as well as in the papers, the stress strain relations are discussed. Furthermore, fracture mechanics and fatigue are also considered as they are of general interest to the project. In Chapter 4, the concept of inverse modelling is outlined. In particular the choice of cost-function, the stress computation and a model for optimisation, not requiring the gradients of the object function, is discussed and used for the parameter estimation. Finally, in Chapter 5 some concluding remarks and future work is discussed. The thesis also comprises two appended papers. The first paper discusses the influence on parameters and estimated life dependent on the choice of cost-function in the optimisation process. In the second a method for stress integration is presented and discussed.

2. Gas turbine engines

The work presented here is being carried out mainly at Volvo Aero Corporation and the methodology developed in this work will be applied in design of components w.r.t. high temperatures and fatigue. A brief introduction to the jet engine and its environment is given below. A more detailed description of gas turbine propulsion can be found in Mattingly [2].

Jet propulsion systems can be subdivided into two categories: Air-breathing and non-air-breathing systems. The air breathing propulsion systems include the following types of engines, the reciprocating, turbojet, turbofan, ramjet, turboprop and the turboshaft engines. The non-air-breathing engines include, i.e. the rocket engines, nuclear and electric propulsion systems. The non-air-breathing propulsion systems are characterised by the fact that they carry both fuel and oxidiser within the aerospace vehicle.

The idea of air-breathing jet propulsion originated at the beginning of the 20th century. According to Mattingly [2] air-breathing jet propulsion can be defined from a technical standpoint, as a special type of internal combustion engine which produces its net output power as the rate of change in the kinetic energy of the engine's working fluid. A few patents on early air-breathing jet propulsion were presented in the beginning of the 20th century. Lorin as early as in 1908, with an engine based on a piston machinery, Lorin again in 1913, now an engine based on ram compression in supersonic flight, i.e. the ramjet. Later Guillaume patented an engine based on turbomachinery in 1921. All these patent clearly describes the air-breathing jet principles but were not executed in practice. The reason for this lies mainly in the interdependency between aerovehicle and propulsion system. The aerovehicles of the twenties simply had not the capabilities to exploit the jet engine, they were too slow, and the propulsive efficiency of the jet engines at these speeds was very low. This resulted in the jet engine concepts being forgotten for a long time. The first patent of a turbojet engine, which was developed and produced, was that of Sir Frank Whittle in January 1930. The patent shows a multistage, axial-flow compressor followed by a radial compressor, and an exhaust nozzle. Such configurations are still used today for small and medium power output engines. The first flight however of a turbojet aircraft was made on August 27, 1939. It was a Heinkel, He-178, and the German Hans Von Ohain had developed the engine, called He.S3B.

The function of a gas turbine engine is based on a thermodynamic cycle, the so-called Brayton cycle. The Brayton cycle consists of four ideal processes, two constant pressure heat processes together with isentropic compression and expansion. The gas turbine engine is built up of a number of stages; c.f. Figure 1. The core of the gas turbine is the gas generator, which
consist of a compressor, a combustor and a turbine. The purpose of the gas generator is to supply high-temperature and high-pressure gas. Behind the gas generator a nozzle is placed to create the jet stream. These parts are common for the turbojet, turbofan, turboprop and the turboshaft engines. The ramjet engine, however, consists only of an inlet, a combustion zone and a nozzle. For the ramjet, air is compressed in the inlet and then directly enters the combustion zone.

The thrust of a turbojet is developed by compressing air in the inlet and compressor, burning a mix of the air and fuel in the combustor, and expanding the hot gas stream through the turbine and nozzle. The expansion of the gas through the turbine supplies the power to rotate the compressor. For an aircraft part of the gas energy is also used in an additional low-pressure turbine which drives the fan in front of the compressor. The remaining gas stream is used to get the net thrust delivered by the engine.

In turbofan engines, which is the most common type of jet engine, some of the air from the fan is not fed into the core engine but passes through a duct on the outside of the core, cf. Figure 2. This air is used both to provide thrust and cooling metals in sections with high temperatures. This cooling air, taken from the fan or the compressor, allows higher turbine inlet temperatures. But there is an upper limit for how much air can be used for cooling, since this air does not contribute directly to the thrust performance. The mass flow ratio between the two air streams, called the bypass ratio, is usually small (two to three) in a military engine but higher in commercial engines. In more advanced engines the ratio can be as high as six. A high bypass ratio results in high overall efficiency, i.e. long flight range, strong increase in propulsive thrust at low flight speed, lower jet velocity i.e. noise reduction, and lower fuel consumption. The drawback of an engine with high bypass ratio is that it works like a ducted propeller. This means that at speeds approaching the speed of sound, compressibility effects set in and the propeller loses its aerodynamic efficiency. Due to the rotation of the propeller, the propeller tip will approach the speed of sound before the vehicle approaches the speed of sound. On the other hand, a lower by pass ratio results in higher thrust to weight ratio and better efficiency at supersonic velocities. This is the reason military engines have lower by pass ratios than commercial engines. In order to increase the thrust even more, often in military engines, auxiliary power can be generated in an afterburner, c.f. Figure 2. Though raising the thrust to weight ratio results in higher temperatures and stresses on the engine.
Due to the different designs each type of engine will only operate efficiently within a certain range of altitudes and Mach number, c.f. Figure 3. Similar limitations apply to the airframe structure. Figure 3, below, shows the approximate velocities and altitude limits or corridor of flight for the different types of engines. A lift limit, a temperature limit and an aerodynamic force limit bound this corridor.

There has been a tremendous development of the gas turbine engine since it first appeared and the demands and improvements are constantly increasing. Basically, the design philosophy of the future aircraft engines include simplified designs, better performance (higher speed, altitude and greater range), higher efficiency (specific fuel consumption, and increased thrust), lower noise and emissions, lower maintenance cost and good quality (reliability), Bokulish [3]. Increasing thrust and reducing maintenance are most important for military engines, while emissions, noise and specific fuel consumption are important for commercial transport. Increased thermal efficiency of the gas turbine engine is primarily obtained by raising the turbine entry temperature and the overall pressure ratio, i.e. the pressure over the turbine. In current commercial gas turbine engines typical overall pressure ratio is 1:30 and the turbine entry temperature can reach 1500 °C.

The desire to increase the output effect of the engine therefore requires better high temperature material properties, c.f. Holmquist [4]. Many parts have a working temperature almost at their melting point where they are very viscous. The high temperatures cause the material to yield and the elastic strains will be accompanied by plastic strains, therefore thermal barrier and new cooling techniques are applied. This in turn causes larger thermal
gradients and larger thermal strains. Thermal fatigue is often one of the most significant life-limiting factors for cooled, high temperature components, c.f. Sehitoglu [5]. In the last 25-30 years the material capability has increased with about 120 °C, but the effect gained by more efficient cooling was about 500 °C during the same period.

The problem of jet engine design is complex since a high pressure ratio over the fan means that the blades are rotating faster, resulting in increased noise levels and increasing the impact energy of ingested objects such as ice, hail and birds. However, with a large pressure ratio over the compressor, fan blade tip speed can be kept at a lower level to minimise noise. A reduction of weight means better fuel economy and, e.g. in the fan, reduced blade weight would allow for lighter containment system because there is less energy to absorb in a blade-out situation. Due to a reduced weight of the blades a lighter fan disc can be used to hold the fan blade in place. But at higher thrust levels, the fan blade will be travelling faster, and the impact energy requires sufficient containment in order to reduce the damages from blade-out situations. Thus, some of the weight saved on the blades may be lost to increased containment requirements.

These demands on jet engines, that have to be considered by a designer, imply that the simulation tools need to be more efficient and more accurate. Various types of material models for different applications has been developed during the years and made it possible to predict the material behaviour. The success of future engine manufacturing rely much on the advancements in computational software. They allow the evaluation of designs without having to demonstrate them in full scale testing. For example simulation of tests that predict stresses and strains in a component accurately, Bokulish [3]. Also during the manufacturing of the engine parts material models are needed to predict residual stresses and the effects of the manufacturing process, such as welding and cooling. Computer simulations are thus needed to minimise the residual stresses and to decrease the need of costly testing.

3. Material models for high temperature design

The solution of problems in mechanics is based on the equations of equilibrium, geometric compatibility and constitutive models. This thesis focuses on the latter. Development of material models requires observations of micro- and macroscopic effects in the material. The model may be formulated as relations between rates of stress, strain, temperature and some internal variables.

The basis for the different types of model models used in the high temperature regime is given, c.f. Figure 4. Elevated thermomechanical fatigue is one of the most complex topics in material research since it, in general, is a consequence of the combined action of creep, cyclic loads and environment, c.f. Suresh [6]. However, for the mechanical design of jet engines we normally use three main types of models, constitutive equations for the stress-strain relation, a damage model for the crack initiation and a fracture mechanics model for the crack propagation. Before the models can be used a set of material parameters, or constants has to be determined. The estimation of these parameters is normally based on experimental data. In this work, numerical methods are used for the estimation of the model parameters.
3.1 Constitutive relations

In this section we discuss the constitutive equations for stress strain relations. A large amount of literature deals with the description of constitutive modelling under conditions at elevated temperatures, c.f. Eftis & Abdel [8]. As mentioned in the previous section appropriate constitutive models are of utmost importance in thermo-mechanical fatigue. Not only for accurate stress-strain predictions, but also in fracture mechanics modelling, as the crack growth behaviour will be strongly influenced by plastic deformations near the crack tip.

In the following we assume that the total strain $\varepsilon_{ij}$ can be decomposed into the sum of elastic, inelastic, and thermal strain.

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^i + \varepsilon_{ij}^h$$

Where the inelastic strain corresponds to the physical phenomena of:
- Rate independent plasticity
- Viscoplasticity
- Creep

However, for high temperature applications the viscoplastic or creep behaviour is most appropriate. The thermodynamic basis for plasticity and viscoplasticity is treated in detail in some books as Lemaitre & Chaboche [9], and Maugin [10]. Here, viscoplasticity is briefly introduced.

The observable variables are taken as total strain $\varepsilon_{ij}$ and temperature $T$ and the internal variables $\varepsilon_{ij}^i$, $R$, and $\beta_i$ are introduced to account for the certain phenomenon that occurs in the material. In this case the inelastic strain, the isotropic and kinematic hardening. One
objective with the state variable constitutive modelling is to develop a relation between the observable and the internal variables. The existence of such a correlation permits the state of a metal to be determined completely from the current state, i.e. the current value of the internal variables. For dissipative phenomena, such as in viscoplasticity, the state variables in any point are dependent on the deformation history of that point.

The stress can be determined from the generalised hook’s law as
\[ \sigma_{ij} = E_{ijkl} (\dot{e}_{ik} - \dot{e}_{ij}^m) \] (2)
This relation has to be reformulated into an objective rate in case of large deformations.

The relationship between the current microstructure and the previous deformation history, i.e. the history dependence of the material, is included through the hardening parameters in the inelastic flow law. Inelastic deformation occurs at high temperature due to planar slip and dislocation climb, a diffusion controlled mechanism. Formally it can be written as
\[ \dot{\varepsilon}_{ij}^m = \dot{\varepsilon}_{ij}^m (\sigma_{ij}, T, \beta_{ij}, R, \ldots) \] (3)

The state variables, \( \beta_{ij} \), are also called the back stress, which results from the interactions of dislocations with other dislocations, grain boundaries, precipitates and other barriers. The net stress producing slip or inelastic strain is the overstress or the difference between the shear stress and back stress. Thus, hardening is an increase in the resistance to deformation. The net effect of hardening and softening can be separated mathematically into two groups by their orientation dependent or independent characteristics. This establishes the need of two types of hardening state variables. A scalar drag stress is used to model the development of hardening or softening associated with isotropic effects such as point defects, precipitates, cells and subgrains or grains. This isotropic hardening, \( R \), is associated with the cumulative effective inelastic strain \( p \) or the inelastic work \( W \) and can be written in a general form as a hardening function, \( h_1 \), added with an recovery function, \( r_1 \), c.f. Abdel-Kader [11].
\[ \dot{R} = h_1 (R)p \text{or} W - r_1 (R, T) \] (4)
where
\[ \dot{p} = \left\{ \frac{1}{2} \dot{\varepsilon}_{ij}^m \dot{\varepsilon}_{ij}^m \right\}^{\frac{1}{2}} \text{ and } \dot{W} = \int \sigma_{ij} \dot{\varepsilon}_{ij}^m d\tau \] (5)

Recovery \( r_1 \) is a phenomena that is achieved by formation of cells and subgrains. Cells and subgrains are orientation independent and monotonically approach a saturated state under uniform loading conditions.

In addition a back-stress, or kinematic hardening tensor is introduced to model the hardening and recovery effects associated with dislocation pileups. The general form of the kinematic hardening is obtained from Abdel-Kader [11] as
\[ \dot{\beta}_{ij} = h_2 (\beta_{ij}) M_{ij} - r_d (\beta_{ij}, T) N_{ij} - r_s (\beta_{ij}, T) U_{ij} - r_t (\beta_{ij}, T) V_{ij} + \theta (\beta_{ij}, T) F \] (6)

Here, \( h_2, r_d, r_s, r_t, \) and \( \theta \) are functions for the hardening, \( r_d \) the dynamic recovery, \( r_s \) the static thermal recovery and \( \theta \) represents hardening or recovery associated with temperature change. The tensors, \( M_{ij}, N_{ij}, U_{ij}, \) and \( V_{ij} \) are the directional indices associated with the functions \( h_2, r_d, r_s, \) and \( \theta \).
Dynamic recovery can be described as the annihilation of dislocations during deformation that reduces the effective rate of hardening. Temperature and strain-rate dependence results from the balance between the hardening and recovery rates. Static recovery results from the interaction stress between the dislocations themselves. Higher temperatures increase dislocation mobility and promote diffusion. Static recovery lowers the effective steady-hardening during creep.

In this work the model by Bodner & Partom [12] has been used. This is a model without elastic domain and the flow law is simply given as

$$\dot{\varepsilon}^m_{ij} = \dot{\lambda} s_{ij}$$  \hspace{1cm} (7)

The evolution equations of this model are, for the inelastic strain rate

$$\dot{\varepsilon}^m_{ij} = D_0 \frac{s_{ij}}{\sqrt{J_2}} \left[ \frac{1}{2} \left( \frac{Z^t}{J_1} \right)^{1/2} \right]$$  \hspace{1cm} (8)

and for the hardening $Z = Z^t + Z^D$ as a sum of the isotropic and the directional hardening given as

$$\dot{Z}^t = m_1 (Z_1 - Z^t) \dot{\psi} - A_1 Z_1 \left[ \frac{Z^t - Z^2}{Z_1} \right]^{\gamma_1}$$  \hspace{1cm} (9)

$$\dot{\beta}_{ij} = m_2 (\sigma_{ij} - \beta_{ij}) \dot{\psi} - A_2 Z_1 \left[ \frac{\beta_{ij}}{Z_1} \right]^{\gamma_2}$$  \hspace{1cm} (10)

where

$$\sigma_{ij} = \frac{\sigma_{ij}}{J_1}$$  \hspace{1cm} (11)

Here we use the geometric norm $|A_{ij}| = \sqrt{A_{ij} A_{ji}}$. Finally the directional hardening is given as

$$Z^D = \beta_{ij} u_{ij}$$  \hspace{1cm} (12)

The material constants needed for this model is $D_0$, $n$, $m_1$, $m_2$, $Z_1$, $Z_2$, $Z_3$, $r_1$, $r_2$, $A_1$ and $A_2$.

### 3.2 Crack growth

In this section we examine the mechanisms of crack growth and of linear (LEFM) and nonlinear fracture mechanics that are relevant to applications in fatigue. Often modern design approaches to fatigue are based on the premises that the structure is inherently flawed. The fatigue life is then the time for the crack to propagate from an initial assumed crack length to critical dimension. In most metallic material, the failure is preceded by a substantial amount of stable crack propagation under cyclic loading conditions. The rates at which these stable cracks propagate for different combinations of applied stress, crack length and geometrical condition of the cracked structure are of great practical interest.
The growth rate of a fatigue crack is normally expressed in terms of the crack length increment per cycle, \( \frac{da}{dN} \). The values of \( \frac{da}{dN} \) for different loading conditions are determined from experiments. Fatigue crack growth at elevated temperatures can be approached in some different ways. If the conditions are considered linear-elastic or small scale yielding can be assumed we use \( \frac{da}{dN} \) versus the stress intensity factor range \( \Delta K \). For linear fracture mechanics the crack growth can be characterised by the stress intensity factors \( K = K(\sigma, \sqrt{\pi a}) \), which are measures of the intensity of the near-tip field. There exists an annular zone ahead of the crack tip, known as the region of K-dominance, within which the stress intensity factor provides a unique measure of the intensity of stress strain or deformation, c.f. Suresh [6].

Under cyclic loading, and for conditions where the nonlinear zone at the crack tip is a mere perturbation in an otherwise elastic material the onset of crack growth can occur according to the "law" of Paris [13]. Paris showed that the crack growth increment \( \frac{da}{dN} \) is related to the stress intensity factor range, by the power law relationship

\[
\frac{da}{dN} = C(\Delta K)^m
\]

where \( C \) and \( m \) are constants. The simplicity of Paris law provides a means of estimating the useful life of fatigued component for design or failure analysis. For cyclic variation of the stress field, the linear elastic fracture mechanics of the rate of fatigue crack growth should be based on the stress intensity factor range. Thus, given as

\[
\Delta K = K_{\text{max}} - K_{\text{min}} = Y\Delta\sigma\sqrt{\pi a}
\]

where

\[
\Delta\sigma = \sigma_{\text{max}} - \sigma_{\text{min}}
\]

\( \sigma \) is the minimum and maximum of the fatigue stress and \( Y \) are a geometrical factor dependent on the ratio of crack length \( a \) to the width of the specimen.

The fatigue life or number of cycles to failure is calculated by integrating \( \frac{da}{dN} \) from an assumed initial crack length \( a_0 \) to critical \( a_c \). Many semi-empirical models for fatigue crack growth have been proposed in the literature to account for the load ratio dependence of \( \frac{da}{dN} \) and the deviation of crack growth behaviour in the near-threshold and final failure regimes, c.f. Saxena [14]. Among these models, the most widely used in industrial applications are the empirical approaches of modified Forman [15] and Walker [16]. The modified Forman model is a modification of an earlier proposed crack growth relation. The modified equation fairly simple and accounts for the influence of R-ration, thresholds, and increasing \( \frac{da}{dN} \) as \( K_{\text{max}} \) approaches \( K_c \). It is described by the following equation

\[
\frac{da}{dN} = C(1 - R)^n \Delta K^m \left[ \Delta K - (1 - C_0 R)^n \Delta K_0 \right]^{p+q} \left[ (1 - R)K_c - \Delta K \right]^q
\]

where \( C_0, c, d, m, n, p \) and \( q \) are material constants, \( K_c \) is the fracture toughness and \( K_0 \) is the threshold under which the crack either remains dormant or grow at extremely low rates. All the above material parameters are determined from experiments.

Fatigue crack growth under large-scale plasticity is based on the J-integral approach and can be used for thermal-mechanical fatigue. The J-integral is defined as

\[
J = \int W d\gamma - T \frac{\partial u}{\partial x} ds \quad \text{where} \quad W = \int_0^L \sigma_{ij} d\xi
\]
is the strain energy density, $T$, is the traction vector and $u$ is the displacement vector normal to the closed path $\Gamma$. Note, that $W$ represents the strain energy density here and not the inelastic work as in the previous chapter. In this formulation we need some constitutive relation for deriving the stress distribution. The corresponding parameters for characterisation of the crack growth under conditions of creep crack growth are $C^\ast$, $C$ and $C(t)$, c.f. Saxena [14]. For creep crack growth the crack increment is obtained per unit of time $\text{da}/\text{dt}$ instead of cycles. LEFM based crack propagation models are the most widely used method in the industry like the aerospace and power generation.

### 3.3 Fatigue damage and crack initiation

Phenomenological approaches are widely used for characterisation of the total fatigue life as a function of stress range, strain range, mean stress and environment. The stress life approach began with the work by Whöler in the 1860s. From this other approaches have evolved such as the equation proposed by Basquin [17] in 1910. He observed that if the number of load reversals to failure plotted as a function of the stress amplitude is a linear relationship on a log-log scale. In this expression the stress amplitude in fully reversed, constant amplitude fatigue test is related to the number of load cycles as

$$\sigma' = \frac{\Delta \sigma}{2} = \sigma'_f (2N_f)^b$$

(18)

$\sigma'_f$ and $b$ are materials dependent parameters. Coffin [18] and Manson [19] introduced later, in 1954, a relationship for plastic strain based fatigue life. They noted that the plastic strain amplitude plotted against the logarithm of the number of load cycles to failure is also a linear relationship. This together with the equation of Basquin, above, resulted in the so-called Coffin-Manson equation

$$\frac{\Delta \varepsilon}{2} = \sigma'_f (2N_f)^b + \varepsilon'_f (2N_f)^c$$

(19)

and later Coffin proposed an extension to cyclic frequency on elevated temperature response. Then Equation (19) becomes,

$$\Delta \varepsilon = c_{f1} (N_f u_{s1})^{\beta_1} + c_{f2} (N_f u_{s2})^{\beta_2}$$

(20)

In the equations above $\sigma'_f, \varepsilon'_f, \sigma, \varepsilon, c_{f1}, c_{f2}, u_s, k, k_s, \beta_1$ and $\beta_2$ are material parameters dependent on the material ductility and the frequency of the loading.

Numerous approaches have been developed during the years to try to predict the total failure life in material subjected to high temperature, c.f. Halford [20]. The simplest and most common approach to predict fatigue life involves the linear damage summation model.

$$\sum \frac{n_i}{N_i} + \sum \frac{f_j}{t_{ji}} = d$$

(21)

where $d$ is the accumulated damage, which often is taken to unity when the life is consumed.

If the material is subjected to $n$ cycles of stress amplitude $\Delta \sigma$ and the number of cycles to failure at this stress amplitude is $N$, the fraction of fatigue damage is given by the first term in Equation (21). This relation is the so-called Palmgren-Miner cumulative damage rule.
Palmgren [21] and Miner [22]. Similarly, with \( t_j \) as the average time at an average stress \( \sigma_j \) and the time to rupture at that stress level is \( t_{Rj} \), the second term in Equation (21), denotes the accumulated creep damage.

4. Inverse modelling

The calibration, i.e. determining of parameters in the material model, with experimental data has to be performed for all the models described in chapter 3.1-3.3. The aim of this section is to outline the procedure of parameter determination for material models in general. For the estimation of the parameters, it is sometimes desirable to use the concept of inverse modelling. The inverse problem can be described as in as follows, c.f. Figure 5, and Salomonsson [23].

Normally, an analysis is performed by using a given design specifications, e.g. model parameters, and the result or behaviour is computed. This is usually called solving the direct problem. If we instead are given the desired behaviour or result, the problem becomes to find the design specifications. Thus, the inverse problem is the direct problem reversed. Usually the result of some measurement of observable parameters is used to infer the actual values of the model parameters.

Mathematically the inverse problem of finding material parameters for a model can be written as in Mahnken [25]. Figure 7, shows the general idea for parameter estimation. For the direct problem let \( Y \) be the parameter space, \( I = [0, \tau] \) the time interval of interest, \( U \) the computed observation space, and finally \( P \) denotes any measured material point. Then \( U \times I \) is the solution space for e.g. the strain and stresses in a mechanical problem of the former section. The direct problem can be stated as

\[
\kappa_i \Rightarrow \sigma_i(\kappa_i) \quad \kappa \in Y
\]  

(22)

where \( \kappa_i \) are the internal variables, i.e. the material parameters, in an strain-controlled problem. This can be seen as the mapping of set \( Y \) into set \( U \times I \). Where every element of \( U \times I \) is the image of at least one element of \( Y \) and can be specified by the solution operator

\[
S : \left\{ \begin{array}{l} \kappa_i \Rightarrow \sigma_i(\kappa_i) \\ Y \rightarrow U \times I \end{array} \right. 
\]  

(23)
which describes the direct problem. Another set of interest is the reference data from experiments. Let $\tilde{u}_i \in P \times I$ symbolise this set of data and $u_i \in U \times I$ be the computed values. The intention here is to estimate the parameters $\kappa_i$ so that $U \times I$ matches $P \times I$. It is now tempting to formulate the inverse problem by simply reversing expression (23) by backward calculation. This is however not possible mainly due to the fact that

1. $U \times I \neq P \times I$. Thus it is not possible to map $Y$ into $P \times I$.
2. There exists no unique inverse operator $S$.

The first item implies that no mathematical model exactly can describe the real material behaviour. In order to circumvent the above problem a optimisation method is applied in which it is searched for the closest possible match of $\sigma_f(\kappa'_i)$ and $\tilde{u}_i$. This corresponds to finding the parameters $\kappa_i$ that minimises the functional

$$f(\kappa_i) = \| u_f(\kappa'_i) - \tilde{u}_i \|_{U \times I}$$

(24)

The choice of norm on $U \times I$ is essential and may affect the solution as will be discussed further in section 4.2.

![Figure 6. General flow chart over an inverse method for parameter determination.](image)

Figure 6, show the different parts in the estimation process. First the direct problem is solved, in this case the stress computation. The discrepancy between the solution of direct problem and the experiment is evaluated by the cost-function. If the problem has not converged a minimisation is performed, often by optimisation. There is a wide range of issues associated with the inverse problem, discussed by e.g. Gavrus [26], Mahnken [27].

- Choice of suitable cost function.
- Choice of an efficient and reliable optimisation strategy.
• Choice of sensitivity analysis method.
• Model accuracy and effects of discretization errors, i.e. error in stress computation.
• Errors in the experimental data.

In the current work we have focused on the choice of cost-function and the stress computation. There has been a lot of work done in the field of inverse modelling theory and it is well described in some books written by e.g. Norton [28], Tarantola [29] and Bard [30]. Much of the work has been done and is still being performed within the field of automatic control systems for e.g. by Walter & Pronzato [31]. Other areas are e.g. mathematics by Banks [32] and Structural Optimisation and acoustics by Tinnsten [24]. However, some work has been done in estimation of material parameters for mechanical material models and is discussed generally by e.g. Bui [33] and Cailletaud & Pilvin [34]. Many others are involved in the area of constitutive parameter estimation, in various applications. For example, for viscoplastic material models Mahnken & Stein [25], and Olszewski, Sievert & Bertram [35], for metal forming Gavrus, Massoni & Chenot [26], Gelin & Ghouati [36], casting processes Boehmer & Fett [37], hot isostatic pressing Svoboda, Björk & Häggblad [38] and finally estimation from crack propagation tests Hasan, Piva & Viola [39].

4.1 Optimisation

As mentioned above, inverse problems are associated with the minimisation of a cost function. The cost function measures the discrepancy between numerical model predictions and the measured experimental response. The optimisation corresponds to seeking minimum of the cost function subject to constraints, c.f. Gill [40], Nash & Sofer [41], and Achtziger [42].

The optimisation algorithms used to minimise the object function are usually divided into two categories, gradient methods and non-derivative methods. In the latter only the value of the function is needed while, as the name reflect, an expression for the gradient of the object function is needed in the gradient method. In this work we have use a non-derivative method, by Rowan [43].

This method is based on the simplex method by Nelder and Mead [44]. The Simplex method is based on searching for extremes points via search patterns and are called function comparison methods or polytope methods. Searching the full pattern usually requires many function evaluations which make these methods suffer from efficiency disadvantages when the number of design variables increase. However, the method used here divides the function space into subspaces and thereby speeds up the optimisation.

The polytope methods are based on an initial design of n+1, where n is the number of design variables. A geometric figure built up by n+1 points in an n-dimensional space is called a simplex. The corners of this figure are called vertices. The next step is to determine the largest function value among the trials. This vertex is then reflected in the centroid of the other vertices. The process is sequentially forming new simplex figures. The procedure ends when the convergence criteria is satisfied, c.f. Figure 7.
Figure 7. The simplex method of Nelder and Mead applied to a quadratic function, (Achtziger [42])

The typical behaviour of the simplex method applied to a quadratic function is shown in Figure 7. The numbers in the figure refer to the order of approved points during the minimisation. The initial three starting points (0,1,2), c.f. Figure 7, are given with their associated function values $F_0$, $F_1$ and $F_2$. These values span a simplex in function space. Next, the simplex is reflected about the point with the largest functional value once, to the point $F_n$. If this step is successful, i.e. $F_n$ is less than at least one of the other function values, but not the least, this point is chosen as the next valid point. On the other hand, if the new point has the least functional value another step of some length is taken, i.e. to point 3 in the figure above. If however, the new point have a greater function value $F_n$ than all the other function values, the simplex is considered to large and should be contracted. This procedure is then repeated until the function value of the final point meets the convergence criteria.

4.2 Cost-functions

The determination of parameters starts with a definition of a cost function. Usually when fitting model functions to data, the cost function is the difference between computed and experimental results for a set of operating conditions. In this respect often some sort of least squares functional is minimised in order to provide the best agreement between experimental data and model prediction data. The least-squares method is one of the oldest and most widely used estimation procedure, c.f. Gill [40]. Part of its popularity is due to the fact that it can be applied in an ad hoc manner directly to the deterministic model with little concern to the probability distribution of the observations.

A more general formulation of the cost-function can be to formulate the cost-function as functional which evaluates, for a given parameter set, the distance between the model prediction and the observations, see i.e. Cailletaud [34]. This functional can be written

$$ \theta(Y,Y^*) = \int_{t=0}^{t_{max}} \left\| Y'(t) - Y(\alpha,t) \right\|_p W(t) dt $$

(25)

where $Y^*$ is the experiment, $Y$ is the simulation and $W$ is a weight function. They are all functions of time $t$, but the simulation is also a function of the internal variables $\alpha$.

Occasionally, some part of the experiment is more reliable or more relevant to the future application of the constitutive model. It is then possible to put an emphasis to these parts of the test through the weight function. It is usually difficult to take into account the behaviour of the material in the identification process where the model is not adequate. The way to
compensate for this for a given application is to use a suitable cost-function taking these effects into account, see also Paper A.

4.3 Stress integration

The choice of formulation and computational model of the elasto-plastic behaviour determine the quality of the simulations. It is most important to compute the stresses both accurately and efficient since an error introduced in the stress integration can not be compensated for later in the simulations. The stress integration is therefore also important for the accuracy of the material parameter estimation. The integration of the constitutive equations is normally performed with some type of return mapping algorithm. A common method for stress integration is the so-called effective stress function method, initially introduced by Kojić & Bathe [45]. An extension of this method has been used in paper B.

5. Summary of papers

This section contains a brief summary of the two appended papers and their relation in the thesis.

5.1 Paper A

In Paper A, the choice of cost-function for the stress prediction in life prediction process is discussed. This cost-function is particularly important when the material model can not represent the material behaviour properly. Three different cost-functions are presented and used for evaluation of an experiment. The result shows that the obtained parameters differ much from each other dependent on the choice of cost-function. This is expected since the model used can not describe the material behaviour in the test. The different results show that it is important to consider the application that the model will be used in when choosing a cost-function.

5.2 Paper B

In Paper B, a method for integration of constitutive equations is presented. This method is an extension to the Effective stress function by Kojić & Bathe [45]. The extension consists of an additional term for recovery in the transverse direction to the direction of inelastic flow. The approach is a projection with subsequent embedding of the tensor entities on the direction of the current inelastic strain. This enables an evaluation of a scalar expression instead of having to solve the equations in multiaxial form. The relation to the thesis is through the importance of an accurate method for stress integration. This is because an error introduced in the stress integration can not be compensated for later in the simulations.

6. Discussion and future work

The aim of this work is to deal with the problem of developing a methodology for test evaluation. Estimating parameters in an accurate and efficient way is important in order to perform advanced simulations where complex material models are used. This thesis describes the process of estimating parameters for material models. Issues on the cost-function are the focus in paper A and the stress computation is dealt with in paper B.

There are a lot of possible sources of inaccuracy in the estimation process. It is of interest to get a better understanding of how to evaluate the inaccuracies in the process. Future work will involve inverse modelling of complex experiments such as contact fatigue, using the finite element method. Another important research topic is to quantify the uncertainties in test data and the subsequent uncertainties in the stress-strain behaviour.
References


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Choice of cost-function in material parameter determination leading to variation in stress prediction
Choice of cost-function in material parameter determination leading to variation in stress prediction

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Abstract
In this paper, we show by an example, that the choice of cost function and the weighting of test data in material parameter estimation for constitutive models have a large influence on the obtained material parameters. Thus, it is important to choose a cost function that captures the details in available experiments that are relevant to the application at hand. This is important when the material model can not represent the material behaviour in all details.

Keywords
Life prediction, stress analysis, cost-function, parameter identification

1. Introduction
In order to increase reliability and reduce the risk of engine failure for modern jet engines, advanced mathematical models are increasingly used. For example, being able to accurately predict the stress is crucial in life prediction of high temperature components. Models found in engineering or research problems are generally based on a phenomenological analysis in connection with experimental results. Two types of models are needed in the life prediction analysis. They are a constitutive model for the stress-strain relation and a damage model for the cumulative fatigue.

This paper focuses on the inelastic constitutive model, which may require a large number of parameters. The lack of appropriate material parameters is a major obstacle when doing advanced structural computations. The problem of identification of parameters for advanced mathematical models is a classical engineering and research problem, see e.g. Bard [1]. The identification process must be formulated as a so called inverse problem, leading to the solution of a non-linear optimisation problem. A cost-function to be minimised is formulated. It is based on the deviation between the model simulations and the experimental results. The cost-function can include a weight function if some ranges of the test space are considered more important than others for the application considered.

The objective of this study is to illustrate the influence of the choice of cost-function. This choice is particularly important when the constitutive model can not represent the material behaviour completely. Thus, the parameters have to be identified considering the application at hand. A small deviation in the parameters can cause a relatively large change in stress level and in the end a large deviation in estimated life. The total error estimate in a stress prediction is dependent on the uncertainties in the model as well as in the parameters. The latter is in turn dependent on the way the parameters are obtained, i.e. experimental uncertainties or uncertainties in the optimisation routine. The paper outlines the process of estimating parameters leading to life prediction. The constitutive model with the parameters is applied to a case in order to quantitatively find the relation between cost- and weight functions, material parameters and finally predicted life. It is also possible to use this procedure to determine the effect of errors or other changes in each activity on the predicted life.
2. Process of parameter estimation leading to life prediction

The process of parameter estimation leading to the assessed life consists of a series of linked activities, see Fig. 1. Each activity requires specific models, methods and data depending on the situation. The activities in the middle uses methods (to the right) and data (to the left), see also Isaksson [2].

![Diagram of process leading to life prediction](image)

To achieve an assessed life, models and methods for stress prediction, damage and load profiles are required together with material data. The focus in this paper is on the parameter estimation activity. Although the same reasoning can be applied to all steps in the life prediction chain, here we will focus on the structural analysis box in the life prediction chain. Specifically on how to obtain correct data, given a plasticity model and loading history as calculated in previous activities, c.f. Fig. 1.

2.1 Parameter estimation

There exist various procedures to obtain material parameters, which in the mathematical terminology results in an inverse problem, c.f. Tarantola [3]. Parameters can sometimes be obtained by “hand-fitting”. However, this requires ideal test-conditions with a few parameters, homogeneous stress-strain relation and easily identifiable parameters. Therefore, the parameter estimation is often treated as an optimisation problem.

The parameter estimation problem can be described as shown in Fig. 2. The material response is simulated and the agreement between experimental data and the simulation is measured by a cost function. The cost function $\theta(Y,Y^*)$ is formulated as a functional, which evaluates, for a given parameter set, the difference between the model prediction, $Y$, and the experimental observations, $Y^*$. The latter may need filtering if the test data contains noise due to disturbances. If this difference is not acceptable, a new set of coefficients is chosen according to the optimisation algorithm. Some of the ingredients in the estimation process are discussed in the following.
2.1 Constitutive model
Mathematical models of material behaviour generally express phenomenological variables such as strain, stress or crack length as function or functionals of dependent variables and possibly internal variables. “The analytical shape of functions for inelastic strain and internal variables are chosen on the basis of physical and historical background, on the basis of applicability of the model and authors fancy”, Chaboche [4].

Many useful models involve parameters and internal variables that are not directly measurable, such as hardening parameters in (visco-) plasticity. The internal variables are only indirectly measurable by their effects on observable quantities such as stress, strain rupture, time and crack length. A material model is an idealisation of the real behaviour and thus cannot predict all effects occurring in reality. Here, within the framework of elasto-plasticity we will use the total strain as \( \varepsilon = \varepsilon' + \varepsilon'' \). Specifically we will study \( \varepsilon'' \) given from a power-law equation, see Lemaitre [5]. Thus, the inelastic strain is obtained as

\[
\varepsilon'' = \left( \frac{\sigma}{K} \right)^n
\]  

which together with Hooke’s law \( \sigma = E\varepsilon' \) will give us the total strain as

\[
\varepsilon = \varepsilon' + \left( \frac{\sigma}{K} \right)^n
\]  

The model is chosen deliberately simple in order to illustrate the influence of the cost-function when the model can not represent all phenomena present in the experiment.

2.2 Cost-function
The work presented here addresses the problem of choosing an appropriate cost-function. Occasionally, some part of the experiment is more reliable or more relevant to the future application of the constitutive model. It is then possible to put an emphasis to these parts of the test by assigning a weight function, \( W \). As before mentioned, it is usually difficult to take into account the behaviour of the model in the identification process where the model is not adequate. This may be where large changes in conditions such as strain-rate occur. Especially if this change is a minor part of the total simulation. This may for example cause the effects of relaxation to
dominate at the expense of the high strain-rate effects. The way to compensate for this for a given application is to use a suitable cost-function taking these effects into account.

The cost-function is classically formulated as functional which evaluates, for a given parameter set, the distance between the model prediction and the observations, see i.e. Cailletaud [6]. The functional can be written

$$\theta(Y, Y^*) = \int_{t_0}^{t_f} \left| Y^*(t) - Y(\alpha, t) \right|^p W(t) dt$$  \hspace{1cm} (3).

This general formulation still presents questions, such as how to choose the best type of norm, $p$, to evaluate the functional, the weight function and the observable variables, $Y$. It is also possible to integrate w.r.t. other variables than time, $t$. To study the effect of the choice of cost function, we apply three different cost functions. All are based on the $L^2$-norm squared, i.e. $p=n=2$ in equation (3), but with different integrators. They are

$$\theta(Y, Y^*) = \sum_{n=0}^{\iota} \left| Y^*(t) - Y(\alpha, t) \right|^2 W(t) dt$$ \hspace{1cm} (4)

and

$$\theta(Y, Y^*) = \sum_{n=0}^{\iota} \left| Y^*(t) - Y(\alpha, t) \right|^2 W(\varepsilon) \Delta t$$ \hspace{1cm} (5).

Where $\varepsilon(t)$ is the strain, $\varepsilon(\tau)$ is the inelastic strain and $\alpha$ are the parameters to be found. The absolute values on the integrators $d\varepsilon$ and $d\varepsilon^{\tau}$ arises from the fact that all contributions from the differences in the cost-function should be positive. The time measurement $dt$ is of course always positive.

The cost function in equation (3) with $L^2$-norm is most commonly used, see e.g. Cailletaud [6] and Bard [1]. Expressions (3-5) are in practice evaluated using a finite sum. I.e. the difference between the model simulations and the experimental observations are being evaluated at a finite number of observation times. Thus, equations (3-5) become

$$\theta(Y, Y^*) = \sum_{n=0}^{\iota} \left| Y^*(\tau) - Y(P, \tau) \right|^2 W(\Delta \varepsilon) \Delta \tau$$ \hspace{1cm} (6)

$$\theta(Y, Y^*) = \sum_{n=0}^{\iota} \left| Y^*(\tau) - Y(P, \tau) \right|^2 W(\Delta \varepsilon^{\tau}) \Delta \varepsilon^{\tau}$$ \hspace{1cm} (7)

$$\theta(Y, Y^*) = \sum_{n=0}^{\iota} \left| Y^*(\tau) - Y(P, \tau) \right|^2 W(\varepsilon) \Delta \varepsilon$$ \hspace{1cm} (8)

2.3 Optimisation method

There are many optimisation algorithms, commonly divided into two groups. The gradient methods which require that the gradient of the cost-function can be supplied, and the non-gradient methods which only require that the function value can be computed. In this work, we employ the Subplex optimisation algorithm by Rowan [7]. Subplex is an abbreviation for subspace simplex. This optimisation method is a non-gradient method and is a generalisation of the well-known simplex method by Nelder and Mead [8]. The advantage with a non-gradient method is that it makes no assumptions about the cost-function, such as continuity, and only need the function value of the cost-function. This makes them relatively insensitive to noise in the evaluation function. A problem is however that they often are very inefficient when the dimension of the problem is high, the dimension being equal to the number of parameters to be found. The Subplex method decomposes the problem into low dimensional subspaces. Then Subplex uses the Nelder Mead simplex method to search the subspaces, which can be searched effective. The Subplex method is considered faster and more stable than most non-gradient methods, at least when the dimensions are high.
2.4 Experiment
The test for parameter estimation was performed as a uniaxial tension-relaxation test where a constant strain rate tension was followed by stress relaxation at constant applied strain. The Ti 6-2-4-2 test specimen was circular with diameter 6.34 mm. The test was strain controlled and the force and times were measured during the test.

3. Results

3.1 Parameter estimation
The parameter estimation was performed for five different cases. The two first addressed the weighting of test data. In order to see how the parameters are affected by weighting, the same test have been used in both cases, but with different parts weighted. In both cases, the cost-function from equation (6) was used with a stepwise constant weight function \( W \). In the first test the weight function was defined as

\[
W_1 = \begin{cases} 10^4 & t \leq 2.4s \\ 1 & t > 2.4s \end{cases} \quad (9).
\]

For the second test the used weight function was

\[
W_2 = \begin{cases} 1 & t \leq 2.4s \\ 10^{-4} & t > 2.4s \end{cases} \quad (10)
\]

where \( t \) is the time in the experiment. The parameters for the two cases are presented in Table 1.

A comparison of simulated stress as a function of time is shown in Fig. 3 for the two cases.

Table 1. Parameters from the identification process with varying weight functions.

<table>
<thead>
<tr>
<th>Weight function</th>
<th>E(MPa)</th>
<th>n</th>
<th>K(MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_1 )</td>
<td>84</td>
<td>25.0</td>
<td>697</td>
</tr>
<tr>
<td>( W_2 )</td>
<td>90 000</td>
<td>12.1</td>
<td>500</td>
</tr>
</tbody>
</table>

Next, the identification was performed using cost-functions with different integrators according to equations (6-8). Now, the weight function was set to \( W = 1 \), for all cases. The result from the parameter identification are shown in Table 2. Fig. 3 shows the differences generated by the cost-functions in a stress-time diagram and Fig. 4 in a stress-strain diagram. Note that the identical tension-relaxation test is plotted in both figures but with time and strain on the abscissa, respectively.

Table 2. Parameters from identification with different integrators in the cost-function.

<table>
<thead>
<tr>
<th>Integrator</th>
<th>E(MPa)</th>
<th>N</th>
<th>K(Map)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time ( t )</td>
<td>89909</td>
<td>12.1</td>
<td>500</td>
</tr>
<tr>
<td>Total strain ( \varepsilon )</td>
<td>83569</td>
<td>11.3</td>
<td>1024</td>
</tr>
<tr>
<td>Inelastic strain ( \varepsilon'' )</td>
<td>80000</td>
<td>16.1</td>
<td>885</td>
</tr>
</tbody>
</table>
3.2 Life prediction

To demonstrate the effect of choice of cost-function on life prediction, the LCF-life on an engine component was calculated. A typical mission for a fighter plane was used as load cycle and the life prediction was made in a single point. The material point is considered exposed to uniaxial strain controlled load. Stress as function of time is computed from the load cycles using the power-law constitutive model and the parameters obtained in the inverse modelling analysis. This stress state is then used to compute the life. The Rain-flow count method is used for cycle
counting and Basquin-Coffin-Manson-Morrow law together with Palmgren-Miner rule are used for accumulating damage, Bannantine [9].

The result from the life prediction analysis is shown in Fig. 5 for stress predictions made with two sets of parameters. The first set is obtained from the cost-function (6) i.e. with time as integrator and the second from (7) with strain as integrator. The parameters are shown in Table 2.

![Graph showing predicted life for different integrators in the cost-functions.](image)

**Fig. 5.** Estimated life for different integrators in the cost-functions.

### 4. Discussion

Material models are an idealisation of the material behaviour. In this paper a very simple model has been used to demonstrate the principle. However, similar effects would occur for more complex models, although any reasonable viscoplastic model would give a better fit to this simple test. The choice of cost function becomes even more important when the model can not fit the test. For example, the traditional cost function using time integration fails when phenomena of different time scales are involved. Thus, it is important to choose a cost function that captures the details in available experiments that are as close as possible to the application at hand. This may be almost as important as using a sufficiently good material model and experiments that trigger all relevant mechanisms.

The result of the life prediction analysis demonstrates the importance of a correct set of material parameters. A small change in the parameters will cause a change in the stress state, which in turn will cause large effect on the predicted life. This emphasises the importance of obtaining a correct stress prediction and thus an appropriate set of parameters together with a suitable constitutive model. In the end, this puts requirements on an appropriate choice of cost-function.

An appropriate choice of cost-function is thus an important piece in the procedure for obtaining material parameters adapted for a specific application. Application specific parameters would increase the reliability of stress computations and consequently the life prediction analysis. The long-term goal is to achieve a method for classification of the parameters according to their field of application and their uncertainties and thus obtain a stress prediction with an error estimate.

### Acknowledgement

This work was supported by NFFP, the National Aerospace Research Programme, in Sweden.

### References


Paper B

An Alternative Derivation of the Effective Stress Function Method with an Extension to Transverse Recovery
An Alternative Derivation of the Effective Stress Function Method with an Extension to Transverse Recovery

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Abstract

A method of radial return type for numerical integration of a class of viscoplastic material models is presented. Specifically, for integration methods that enable à priori calculation of the direction of inelastic strain, it is possible to perform the local integration on a scalar equation. We present explicit equations for a projection-embedding approach. This method is used to show how to handle the transverse recovery term introduced by the phenomena of static recovery. This is shown to be straightforward in the projection-embedding method. The transverse recovery of the kinematic hardening originates from recovery in a direction separate from the direction of inelastic flow. The method contains projection of the triaxial state in the inelastic strain direction and a subsequent embedding in triaxial tensor form of the uniaxially integrated stress and inelastic strain increments.

KEY WORDS: Transverse recovery, inelastic deformation, radial return, integration, constitutive model, Bodner-Partom

1. Introduction

In many high temperature applications, such as jet engine components, thermo-mechanical fatigue of materials limits the service life. The demands for improved reliability require an accurate computational prediction. This underline the need for robust, reliable and computer efficient methods of modelling the stress-strain relations. However, the choice of formulation and computational model of the elasto-plastic behaviour determine the quality of the simulations. It is most important to compute the stresses both accurately and efficiently since an error introduced in the stress integration will have severe effects on later steps in the simulation. The inelastic behaviour of the material is characterised by rate dependent constitutive equations and can in general be described as a system of non-linear equations.

Here, we will focus on the description of rate-dependent behaviour within the framework of viscoplasticity. We deal with the issue of efficient integration and iteration for computing stress within a strain-driven format. In particular, a radial return type of method usually referred to as the Effective stress function method (ESF) is derived using a projection-embedding approach (P-E), c.f. Järvstråt [1]. The ESF method was originally proposed by Kojic & Bathe [2] and is based on the radial return method by Wilkins [3]. The radial return method has since been applied to more general materials, such as mixed kinematic-isotropic Von Mises hardening, by, e.g., Simo and Taylor [4] and Ortiz and Popov [5]. Lush [6] generalised the radial return method to non-linear viscoplasticity. To our knowledge no application of the radial return method to a
material model with transverse recovery of the kinematic hardening tensor has been described in the literature.

In the ESF method the purpose is to derive a scalar expression for reducing the computational effort. The ESF method deals with "effective scalar" values while the approach presented here defines an "effective direction" onto which the tensorial variables are projected.

In this work we consider a stress integration methods and derive the ESF method by the concept of projecting stresses and strains on the assumed direction of inelastic deformation. This paves the way for an orthogonal recovery term which is used to improve the algorithm in case of kinematic hardening in a direction different from the inelastic flow. Then, the uniaxial problem can be solved numerically, and finally, the uniaxial result is embedded in the triaxial solution.

The paper is organised as follows: Section 2 contains a brief summary of the basic relation in viscoplasticity. In the following sections 3 and 4, we derive the incremental relations for the chosen model problems, from which updated values of various state variables and the stress can be computed. Here we also show the equivalents of the projection-embedding approach and the ESF method for the model problem. In the latter section we show how static recovery of the kinematic hardening can be handled using the P-E approach.

2. Basic equations

The material behaviour in a metal normally depends on the temperature and deformation histories due to the changing microstructure. This requires the consideration of the microstructure evolution when modelling the material behaviour. Thus, the inelastic response of a material can characterised by rate constitutive equations. In this section we will study a class of rate-dependent inelastic material models.

In order to perform detailed nonlinear simulations the FE, finite element, method can be employed. The general algorithm for a nonlinear simulation is based on fulfilment of globally compatibility and equilibrium of external and internal forces. The residual load, i.e. the force equilibrium, requires an accurate computation of the stresses, which is performed locally. In this paper we will focus on the local problem of stress computation. The FE-analysis is normally of displacement formulation i.e. the displacements are computed in the solution of the global equations. then the local problem of stress computation will be strain driven.

2.1 Equations in viscoplasticity

The constitutive equations that characterise a viscoplastic material, for isothermal conditions, can be briefly stated as follow. The total strain rate obtained from the displacements filed, \( \dot{\varepsilon}_{ij} \), is assumed to be decomposed into an elastic part \( \dot{\varepsilon}_{ij}^e \) and inelastic part \( \dot{\varepsilon}_{ij}^p \) as

\[
\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p
\]  

(1)
Here, and in the following, subscripts denote tensor quantities and notation without subscripts denotes a scalar quantity.

The elastic strain components are given from the generalised Hooke’s law
\[
\dot{\sigma}_{ij} = D_{ijkl} \dot{e}_{ij}^{e}
\]
where \( D_{ijkl} \) is the elastic stiffness tensor and \( \sigma_{ij} \) is the Cauchy stress rate tensor. This has to be reformulated into an objective rate in case of large deformations. The elastic stiffness tensor is assumed temperature dependent and isotropic. Thus, it can be written
\[
D_{ijkl}(\theta) = 2G(\theta)\delta_{ik}\delta_{jl} + \lambda(\theta)\delta_{ij}\delta_{kl}
\]
In Equation (3) \( G(\theta) \) and \( \lambda(\theta) \) are the Lamé’s constants and \( \delta_{ij} \) denotes the second order Kronecker delta.

For viscoplasticity the inelastic strain rate is generally characterised by a set of evolution equations.
\[
\dot{e}_{ij}^{p} = \dot{e}_{ij}^{p}\left(S_{ij}, Z^{j}, \beta_{ij}, e_{ij}^{p}\right)
\]
Here, \( S_{ij} \) is the stress deviator defined
\[
S_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{m}
\]
where the hydrostatic stress is given as
\[
\sigma_{m} = \frac{1}{3} \sigma_{kk}
\]
For simplicity, we restrict ourselves to the case when inelasticity is independent of hydrostatic pressure. However, the extension is straightforward.

Further, the deviatoric strains \( e_{ij} \) are obtained in the same way as the stresses
\[
e_{ij} = e_{ij} - e_{m}\delta_{ij}
\]
with the hydrostatic strain \( e_{m} \) as
\[
e_{m} = \frac{1}{3} e_{kk}
\]
For viscoplasticity the direction of plastic flow is
where the tensor \( \tilde{n}_\gamma \) is a deviator and is defined as being normal to the viscoplastic potential surface. In equation (9) and throughout this section we use the geometric norm \( |A| = \sqrt{\lambda_n \lambda_r} \). The following properties of \( \tilde{n}_\gamma \) apply

\[
\tilde{n}_\gamma = 0 \quad \tilde{n}_\gamma \tilde{n}_\gamma = 1 \quad \text{for} \quad \gamma \in \{1, 2, 3\} \quad \text{(10a,b)}.
\]

The equation for the inelastic strain rate, (4), contains internal variables \( Z^I \) and \( \beta_\gamma \). These variables describe the isotropic and directional hardening of the material and may be written in a general form c.f. Abdel-Kader [7]. The isotropic hardening consists of a hardening function, \( h_1 \), and a softening or recovery function, \( r_1 \)

\[
\dot{Z}^I = h_1 \left( Z^I, \ldots \right) \frac{\dot{\varepsilon}_\nu}{|\dot{\varepsilon}_\nu|} - r_1 \left( Z^I, \ldots \right) \quad \text{(11)}
\]

In turn, the general form of the directional hardening is obtained as

\[
\beta_\gamma = h_2 \left( \beta_\gamma, \ldots \right) \frac{\dot{\varepsilon}_\gamma}{|\dot{\varepsilon}_\gamma|} - r_2 \left( \beta_\gamma, \beta_\gamma, \beta_\gamma, \beta_\gamma, \ldots \right) - r_1 \left( \beta_\gamma, \beta_\gamma, \beta_\gamma, \beta_\gamma, \ldots \right) \quad \text{(12)}
\]

Here, \( h_2 \) are functions for the hardening, \( r_2 \) the dynamic recovery, \( r_1 \) the static recovery and the direction is given by inelastic strain rate and the directional hardening. The transverse recovery is gained in the two latter terms through \( \beta_\gamma \). Recovery is obtained by formation of cells and subgrains. These are oriented independent and monotonically and approach a saturated state under uniform loading conditions. Dynamic recovery can be described as the annihilation of dislocations during deformation that reduces the effective rate of hardening. Temperature and strain-rate dependence results from the balance between the hardening and recovery rates. Static recovery results from the interaction stress between the dislocations themselves. Higher temperatures increase dislocation mobility and promote diffusion.

In order to numerically solve the evolution equations and compute the stress it is desirable to use an integration algorithm which will be computationally efficient and easy to implement into an existing finite element code.

### 3 Numerical implementation

The stress computation algorithm is strain driven as the finite element method is normally implemented as a displacement formulation. The radial return algorithm is an elastic predictor-plastic corrector algorithm. A central concept of the radial return method is using a trial stress that is calculated elastically. The trial stress is projected to the flow surface followed by a plastic corrector phase, so-called return mapping, to obtain the stress at the end of the increment, c.f. Figure 1. Note that although the
The concept is originally conceived for use with rate-independent materials, using the rate dependent flow surface is straightforward.

For viscoplasticity the return mapping is performed in the following manner, referring to Figure 1. The trial state is computed by assuming that the increment from the first time state, A, is elastic. Thus the trial stress $\Delta S^e_i$, is obtained, see point B. The subsequent stress state depends on the amounts of elastic and inelastic strains. The solution is obtained by “guessing” the final state and computing the corresponding inelastic strains in the radial direction $\bar{n}^e$, i.e. the direction of inelastic flow. The process is repeated until the constitutive equations are satisfied.

As mentioned above, the radial return method is particularly attractive because the direction of the plastic strain increment is calculated using only known entities. Thus we simplify the integration of a tensor valued nonlinear system of differential equations into solving a small number of scalar equations.

### 3.1 Derivation of the ESF method for a simple viscoplastic model

In this section the ESF method is derived by using the projection embedding approach. First the tensor quantities stress and strain are being projected on the direction of flow. Then, the constitutive equations are transformed into the solution of scalar system of equations, which easily can be solved numerically. Finally, the uniaxially integrated stress increment and inelastic strain increment are embedded in triaxial tensor space.

The viscoplastic problem is treated as a strain-controlled problem in the sense that strain history is considered prescribed, and the stress is the dependent variable sought. This is the standard procedure for integration of the constitutive model in a FE program. Stress equilibrium is then attained in the global iteration as mentioned before.

Here, we will consider the model by Perzyna without temperature effects, Bathe [7]. The constitutive equations for this model are given in box1, below.
Box 1. The constitutive equations of the Perzyna model.

\[ \dot{\varepsilon} = \gamma S \]

\[ \gamma = \begin{cases} \beta \phi \frac{3}{2\sigma} & \text{if } \sigma > \sigma_0 \\ 0 & \text{if } \sigma \leq \sigma_0 \end{cases} \]

\[ \phi(\sigma) = \left( \frac{\sigma - \sigma_0}{\sigma_0} \right)^\theta \]

In this relation \( \sigma_0, \beta, \theta, N \) are material constants.

### 3.1.1 Discretization of the constitutive equations

The solution is done incrementally and the constitutive equations are evaluated at the integration points of the finite elements. The state at the beginning of the time step and the strain increment is known \( (\varepsilon^{(i)}, \sigma^{(i)}, \Delta \varepsilon, \Delta \sigma) \). The state at the end of the time step is computed from the initial values and the increment in the total strain occurring during the interval. Therefore, it is convenient to write the relations in incremental form.

Given the state correspondent to the state at time \( t \), and using the backward Euler method for integration, we obtain the incremental relations of the stresses and strains from Equation (1)-(9) as

\[ \Delta \varepsilon = \frac{1}{3} \Delta \varepsilon_{ik}, \quad \Delta \varepsilon_0 = \Delta \varepsilon_0 - \Delta \varepsilon_0 \delta_0, \quad (14a,b) \]

\[ \varepsilon^{(i+1)} = \varepsilon^{(i)} + \Delta \varepsilon^{(i+1)} \]

\[ \varepsilon_0^{(i+1)} = \varepsilon_0^{(i)} + \Delta \varepsilon_0^{(i)} \quad (15a,b) \]

\[ \Delta \varepsilon_0^{(i+1)} = \Delta \varepsilon_0^{(i+1)} \]

\[ S_{ij}^{(i+1)} = 3 \varepsilon_{ij}^{(i+1)} (\varepsilon_{ij}^{(i)} + \Delta \varepsilon_{ij}^{(i)} - \Delta \varepsilon_{ij}^{(i)}) \quad (17) \]

\[ \sigma_{ij}^{(i+1)} = D_{ijkl} \left( \varepsilon_{ijkl}^{(i+1)} - \varepsilon_{ijkl}^{(i+1)} \right) \quad (18) \]

where \( G_{ijkl}^{(i+1)} = \frac{E_{ijkl}}{2(1 + \nu)} \quad (19) \)

and for the material model, from Equations (13a-c)
3.1.2 Projection of tensor quantities on the direction of flow

Using the fundamental assumption of radial return procedure, we assume that the flow direction is constant during the time step, and can be determined by the elastically calculated “trial stress”, \( S_{ij}^{\text{trial}} \):

\[
\dddot{\sigma}^{(i+\Delta t)}(\sigma_i^{(i+\Delta t)}) = \left( \frac{\sigma_i^{(i+\Delta t)} - \sigma_0}{\sigma_0} \right)^N
\]

\[
\sigma_{ij}^{(i+\Delta t)} = \frac{\sigma_i^{(i+\Delta t)} + \Delta \sigma_{ij} - \sigma_{ij}^{(i)}}{2}
\]

3.1.3 Solution of scalar equations

Note that Equation (22a) is non-standard although equation (22b) is familiar.

Unfortunately, there is no generally accepted concept for scalar total strain. This is because the elastic and plastic strains in uniaxial test have different tensorial directions (transverse contraction is 0.5 for the incompressible plasticity, but close to 0.3 for elasticity). We therefore define \( \varepsilon^e \) and \( \varepsilon^p \) by the relation

\[
\varepsilon_{ij}^{(i+\Delta t)} \dddot{\bar{n}}_{ij} = \varepsilon_{ij}^{(i+\Delta t)} \dddot{\bar{n}}_{ij} + \frac{1}{3} \dddot{\bar{n}}_{ij} \varepsilon_{ij}^{(i+\Delta t)} + \frac{2}{3} \varepsilon_{ij}^{(i+\Delta t)}
\]

With this definition \( \varepsilon^e \) and \( \varepsilon^p \) can be identified in a uniaxial test as the elastic and inelastic strain, respectively.
If we project the stress tensor in Equation (18), using Equation (3), (15) and (19), we obtain

\[
S^{(i+\Delta t)} = \frac{3}{2} E_{ijkl} \left( e_{ijkl}^{(i+\Delta t)} - e_{ijkl}'^{(i+\Delta t)} \right) = \sqrt{\frac{3}{2} 2G^{(i+\Delta t)} \left( \frac{3}{2} e^{(i+\Delta t)} \Delta e^{(i+\Delta t)} - \frac{3}{2} e^{(i+\Delta t)} \right)}
\]

\[
= 3G^{(i+\Delta t)} (e^{(i+\Delta t)} + \Delta e^{(i+\Delta t)} - e^{(i+\Delta t)} - \Delta e^{(i+\Delta t)})
\]

(24)

In equation (24), the shear modulus appears as the elastic stiffness instead of Young’s modulus, which might be expected by identification with Hooke’s law in scalar form. This is because we are using the conventional definition for inelastic scalar strain rate, which is different from the convention concerning accumulated elastic strain in Hooke’s law. The root of the apparent inconsistency is the different transverse contractions in elastic and inelastic deformation. They cause the relation between isotropic and deviatoric hardening to be different in the elastic and inelastic strain fractions of the uniaxial tests used as scales for the metrics.

The scalar value of the inelastic strain increment is given from Equation (16) as

\[
\Delta e^{(i+\Delta t)} = \frac{2}{3} \Delta \gamma \sigma^{(i+\Delta t)}
\]

(25)

We insert this expression into Equation (24) to obtain

\[
S^{(i+\Delta t)} = 3G^{(i+\Delta t)} (e^{(i)} + \Delta e^{(i+\Delta t)} - e^{(i)} - \frac{2}{3} \Delta \gamma \sigma^{(i+\Delta t)})
\]

(26)

Rearranging and we have

\[
\sigma^{(i+\Delta t)} \left( \frac{1 + v}{E} - \Delta \gamma \right) = \left( e^{(i)} + \Delta e^{(i+\Delta t)} - e^{(i)} - \frac{2}{3} \Delta \gamma \sigma^{(i+\Delta t)} \sigma^{(i)} \right)
\]

(27)

from which the updated stress, \( \sigma^{(i+\Delta t)} \), can be solved numerically by using Equation (20). By squaring both sides of equation (27) and setting

\[
a_\gamma = \left( \frac{1 + v}{E} - \Delta \gamma \right) \quad \text{and} \quad e^{(i+\Delta t)} = e^{(i)} + \Delta e - e^{(i+\Delta t)}
\]

(28)

we end up with

\[
\sigma^{(i+\Delta t)}^2 a_\gamma = \frac{9}{4} e^{(i+\Delta t)} e^{(i+\Delta t)} \sigma^{(i+\Delta t)} \Delta \gamma \left( \sigma^{(i+\Delta t)} \right)^2 = 3 e^{(i+\Delta t)} \Delta \gamma \sigma^{(i+\Delta t)}
\]

(29)

If we use Equation (22d) and the fact that

\[
e^{(i+\Delta t)} = \sqrt{\frac{2}{3}} e^{(i+\Delta t)} \sigma^{(i+\Delta t)}
\]

(30)

inserted into Equation (29), render the exact expression for the effective stress function (ESF) according to Bathe [7], i.e.
\[
f(\sigma^{(t+\Delta t)}) = a_f^2 (\sigma^{(t+\Delta t)})^2 + b \gamma^{(t+\Delta t)} - c^2 \gamma^{(t+\Delta t)} - d^2
\]

\[
a_f = a_f + \Delta t \gamma^{(t+\Delta t)}
\]
\[
b = 3 \Delta t \epsilon_e^{(t+\Delta t)}  
\]
\[
c = \Delta t \sigma^{(t+\Delta t)}
\]
\[
d^2 = \frac{3}{2} e_0^{(t+\Delta t)} - e_0^{(t+\Delta t)}
\]
\[
a_f = \frac{1 + c}{E}
\]

where the object is to solve the equation.

\[
f(\sigma^{(t+\Delta t)}) = 0
\]

Consequently the two approaches are equivalent.

### 3.1.4 Embedding of scalar values into tensor quantities

The current stress and inelastic strain is calculated from the uniaxial material model, i.e. Equations (20) and (27). Subsequently, it is necessary to embed the uniaxial solution into the triaxial stress/strain state, to obtain the required tensor entities. The embedding is the opposite of the projection, we have a direction and a magnitude and the embedding will give us a tensor. For the inelastic strain rate, this is straightforward. The inelastic strain rate is embedded in the direction of the flow.

\[
e_0^{(t+\Delta t)} = e_0^{(t)} + \sqrt{\frac{3}{2}} \Delta t \epsilon_e^{(t+\Delta t)} \hat{n}_0^{(t+\Delta t)}
\]

The stress, however, can not be easily embedded as an on incremental form due to the hydrostatic component. Thus, the easiest way is to use Hooke's law, i.e. Equation (18) directly by inserting Equation (33) and the known updated total strain as

\[
\sigma^{(t+\Delta t)} = D_{ijkl} \left( \epsilon_e^{(t+\Delta t)} - \epsilon_0^{(t+\Delta t)} \right)
\]

Now we have completed the integration for a time step and obtained the updated stress and inelastic strain. Next we will apply this formulation to a more complex material model

### 3.2. Derivation of an ESF function for directional hardening with transverse static recovery

Here, we will address integration of a material model with deviatoric hardening and static recovery, specifically treating the recovery of hardening in a direction different than the current loading. The derivation is straightforward using the projection-embedding approach, although one scalar equation in the transverse direction must be added. The four steps of the projection-embedding approach illustrated above will be followed: 1: Discretization, 2: Projection, 3: Solution of scalar equations, and 4: Embedding.
As an example, we will use the unified viscoplastic model by Bodner [7], given in box 2. In the original equations, the stress is used rather than the stress deviator, but, since the stress is always projected on a deviatoric direction, only the deviatoric part of the stress will affect the inelastic equations.

\[
\varepsilon_{ij}^\nu = \sqrt{2} D_{ij} \exp \left[ -\frac{1}{2} \left( \frac{2(Z' + Z^0)}{3|\mathbf{S}|} \right)^2 \right] \beta_{ij} \\
Z' = m_1(Z_1 - Z_1^0)|\mathbf{S}| \varepsilon_{ij}^\nu - A_1 Z_1 \left( \frac{Z_1' - Z_1^0}{Z_1} \right)^\nu \\
\beta_{ij} = m_2(Z_1, \hat{n}_{ij} - \beta_{ij}) S_{ij} \varepsilon_{ij}^\nu - A_1 Z_1 \left[ \beta_{ij} \right]^{\nu} \frac{\beta_{ij}}{|\mathbf{S}|} \\
\sigma_{ij} = D_{ij} (\varepsilon_{ij} - \varepsilon_{ij}^\nu) \\
Z_0 = \beta_{ij} \hat{n}_{ij} \\
\gamma_{ij} = \frac{S_{ij}}{|\mathbf{S}|}
\]

Box 2. Bodner constitutive equations

Here, \( Z' \) and \( \beta_{ij} \) are the isotropic and deviatoric hardening parameters, respectively. Of the eleven material constants, two (\( D_{ij} \) and \( n \)) are employed in the direct rate equation for strain evolution. Two variables (\( Z_1 \) and \( m_1 \)) describe the isotropic hardening including dynamic recovery, while another two (\( Z_1^0 \) and \( m_2 \)) describe the deviatoric hardening and dynamic recovery. The remaining five constants are devoted to static recovery; (\( A_1 \) and \( r_j \)) to recovery of isotropic hardening and (\( A_1 \), \( Z_1 \) and \( r_j \)) to recovery of deviatoric hardening.

### 3.2.1 Discretization

For the fully implicit Euler method, we simply replace all rate terms with their increments, and all time dependent variables with their values at the end of the time step \( t + \Delta t \) according to section 3.1.1.
\[ \Delta \rho^p = \sqrt{2D_0} \exp \left[ -\frac{1}{2} \left( \frac{2(Z^{(i+\Delta t)} + Z^{D(i+\Delta t)})^2}{3| \mathbf{s}^{(i+\Delta t)} |^2} \right)^{\frac{1}{2}} \right] n_y \Delta t \]

\[ \Delta Z^i = m_i \left( Z_1 - Z^{(i+\Delta t)} \right) S^{(i+\Delta t)} \Delta \rho^p - A_i Z_1 \left( \frac{Z^{(i+\Delta t)} - Z_1}{Z_1} \right) \Delta t \]

\[ \Delta \beta^i_j = m_i \left( Z_1 - Z^{(i+\Delta t)} \right) S^{(i+\Delta t)} \Delta \rho^p - A_i Z_1 \left( \frac{\beta^{(i+\Delta t)} - \beta^i_{ij}}{\beta_{ij}} \right) \Delta t \]

\[ \sigma^i_j = D_{ij} \left( \epsilon^{(i+\Delta t)} - \epsilon^{(i+\Delta t)} \right) \]

\[ Z^{(i+\Delta t)} = \beta_j^{(i+\Delta t)} n_{ij} \]

\[ n_y = \frac{S_y^{(i+\Delta t)}}{|S|_{ij}} \]

### 3.2.2 Projection

In order to reduce system (37) to scalar equations, we will project all variables in the flow direction, and identify the corresponding scalar values. Using the fundamental assumption of radial return procedure, we assume that the flow direction is constant during the time step, and can be determined by the elastically calculated "trial stress". For the Bodner model, this gives

\[ \frac{S_{ij}^{(i+\Delta t)}}{|S|_{ij}} = \frac{S_{ij}^{(i+\Delta t)} + D_{ij} \Delta \epsilon_{ij}}{S_{ij}^{(i+\Delta t)} + D_{ij} \Delta \epsilon_{ij}} = \Delta \epsilon_{ij} \frac{S_{ij}^{(i+\Delta t)}}{|S|_{ij}} \]

Now, projecting all tensorial entities of system 37 on this direction, we define the projections

\[ \epsilon^{(i)} = \sqrt{3} e_y^{(i)} n_y^{(i+\Delta t)} \]

\[ \Delta \epsilon = \sqrt{3} \Delta \epsilon_y n_y^{(i+\Delta t)} \]

\[ \epsilon^{(i+\Delta t)} = \epsilon^{(i)} + \Delta \epsilon \]

\[ \epsilon^{(i)} = \sqrt{3} e_y^{(i)} n_y^{(i+\Delta t)} \]

\[ \epsilon^{(i+\Delta t)} = \epsilon^{(i)} + \Delta \epsilon^p \]

\[ Z^{(i)} = \beta_j^{(i)} n_{ij} \]

\[ Z^{(i+\Delta t)} = Z^{(i)} + \Delta Z^D \]

\[ S^{(i+\Delta t)} = \sqrt{3} S_{ij}^{(i+\Delta t)} n_y^{(i+\Delta t)} \]

Note that in general, the direction of flow is different in different time steps, so that the value of the projected variables differs from the end of one time step to the beginning of the next. Now, according to the equalities (38), all tensor quantities in system (37) may be replaced by these variables, except the directional hardening. This is because equation (37c) may contain a component orthogonal to this direction in the last term, due to previous hardening in other directions than the currently applied stress. Thus we will need a second unit tensor \( \overline{n}_y \), orthogonal to the flow direction \( \overline{n}_y \), to completely
describe the evolution of directional hardening. We define $\pi^\perp_\eta$ as the direction of $\beta^\perp_\eta$, given by the difference between $\beta_\eta$ and its projection on the flow direction (cf., Figure 2). At time $t + \Delta t$, that is:

$$\beta^{\perp(i+\Delta t)}_\eta = \beta^{(i+\Delta t)}_\eta - \beta^{(i+\Delta t)}_\eta \pi^{(i+\Delta t)}_\eta \pi^{(i+\Delta t)}_\eta \pi^{(i+\Delta t)}_\eta$$

and

$$\beta^{(i+\Delta t)}_\eta$$

Figure 2. Direction of projection and transverse recovery

With the transverse direction thus given, only one additional scalar quantity, $Z^T$, is needed to describe the deviatoric hardening:

$$Z^{T(i)} = \beta^i_\eta \pi^\perp_\eta, \Delta Z^T = \Delta \beta_\eta \pi^\perp_\eta, Z^{T(i+\Delta t)} = Z^{T(i)} + \Delta Z^T$$

### 3.2.3 Scalar equations

Projecting equation (37a) on $\vec{n}_\eta$, projecting equation (37e) on both the directions $\vec{n}_\eta$ and $\vec{n}^\perp_\eta$, and replacing all tensorial quantities according to equations (39) and (41), we obtain
\[ \Delta e^p = \frac{2}{\sqrt{3}} D_0 \exp \left[ -\frac{1}{2} \left( \frac{\left( Z^{(i+\Delta)} + Z^{D(\Delta+\Delta)} \right)^2}{s^{(i+\Delta)}} \right)^n \right] \Delta t \]

\[ \Delta Z^I = m_z \left( Z_1 - Z^{(i+\Delta)} \right) S^{(i+\Delta)} \Delta e^p - A_z Z_1 \left[ \frac{Z^{(i+\Delta)}}{Z_1} \right] \Delta t \]

\[ \Delta Z^D = m_z \left( Z_3 - Z^{D(\Delta+\Delta)} \right) S^{(i+\Delta)} \Delta e^p - A_z Z_1 \left[ \frac{\beta^{(i+\Delta)}}{\beta^{D(\Delta+\Delta)}} \right] \Delta t \]

\[ \Delta Z^T = -m_z Z^T^{D(\Delta+\Delta)} S^{(i+\Delta)} \Delta e^p - A_z Z_1 \left[ \frac{\beta^{(i+\Delta)}}{\beta^{D(\Delta+\Delta)}} \right] \Delta t \]

\[ S^{(i+\Delta)} = 3G \left( e^{(i+\Delta)} - e^{D(\Delta+\Delta)} \right) \]

\[ \left| \beta^{(i+\Delta)} \right| = \sqrt{\left( Z^{D(\Delta+\Delta)} \right)^2 + \left( Z^T^{(i+\Delta)} \right)^2} \]

The system (42) is a non-linear equation system with five equations for the five unknowns \( \Delta e^p, \Delta Z^I, \Delta Z^D, \Delta Z^T \) and \( S^{(i+\Delta)} \). It is solved numerically.

### 3.2.4 Embedding

Since all inelasticity in the increment occur in one direction for the radial return approximation the inelastic strain increment \( \Delta e^p \), the embedding is simply a multiplication with the flow direction.

\[ e^{p(i+\Delta)} = e^{p(i)} + \sqrt{\frac{1}{2}} \Delta e^p \bar{n}^{(i+\Delta)} \]  \( (43) \)

The isotropic hardening \( Z^I \) is already a scalar and easily updated. The directional hardening is divided in the two components defined above, and the two tensors must be added when updating the directional hardening tensor:

\[ \Delta \beta = \left( \Delta Z^T \bar{n}_{\beta}^{(i+\Delta)} + Z^D \bar{n}_{\beta}^{(i+\Delta)} \right) \]  \( (44) \)

The stress would be more cumbersome to embed as an increment because of the hydrostatic stress component. However, since we have derived the inelastic strain, we can apply Hooke’s law directly:

\[ \sigma^{(i+\Delta)} = D_{\delta l} \left( e^{(i+\Delta)} - e^{D(\Delta+\Delta)} \right) \]  \( (45) \)

The complete integration procedure consists of projection according to equations (39) and (41), numerical solution of the scalar equation system (42), and finally embedding according to equations (43), (44) and (45).
4. Concluding Remarks

A method based on the radial return integration method has been presented for a general class of viscoplastic material models, and specifically the Bodner model. We have shown that projection on the direction of flow can simplify the implementation of material models, because only a scalar function needs to be integrated. This is equivalent to a fully multiaxial method.

An important consideration for the integration is that the integration method considers recovery even if it may be in a direction different from the direction of inelastic flow. This is difficult to treat with the conventional ESF approach as two scalars are needed for the kinematic hardening. Omitting this transverse recovery term may cause an error in the integration. Here, we show how the problem of transverse recovery can be handled with the method of projection and embedding.

Acknowledgements

This work was supported by NFFP, the National Aerospace Research Programme, in Sweden.

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